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APPENDIX. THE RANGE OF THE EXCHANGE POTENTIAL

There has been some uncertainty as to what quantity should properly be called the range of the exchange potential in the case of the scattering of two unequal mass particles, such as πN scattering. The discussion in Sec. II clarifies this situation.

It will be seen from expressions (2.12) and (2.13) that the absorptive parts in the t and u channels having the same value of the integration variable x' superimpose each other. Now $x'=t$ for t absorptive parts and $x'=u-(m^2-1)^2/s$ for u absorptive parts. Hence the range of the exchange force arising from the exchange of mass $\sqrt{u}$ is $[u-(m^2-1)^2/s]^{-1/2}$ in the sense that $(t)^{-\frac{1}{2}}$ is the range of the direct force arising from an exchange of mass $\sqrt{t}$ in the t channel. Unlike the direct force, the range of the exchange force is energy dependent and gets smaller as the energy gets larger. In particular, the exchange of a single nucleon gives rise at low energies to a force of range of approximately $(2m)^{-1/2}$ and approaches the naively expected range $(m)^{-1}$ only at very high energy.

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Fluctuation Compressibility Theorem and Its Application to the Pairing Model

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A theorem of statistical mechanics relates density fluctuations to compressibility. A new derivation of this is given. The theorem is violated in the BCS model of a superconductor. The difficulty is resolved by those same improvements in the theory which lead to a gauge-invariant Meissner effect.

I. INTRODUCTION

It has been observed by Lüders[1] that density fluctuations in the BCS model of a superconductor violate a standard result of statistical mechanics (Sec. 2, Sec. 3). The difficulty is analyzed here. It is found to be resolved, at least for zero temperature, by those same improvements of the theory which lead to a gauge-invariant Meissner effect (Sec. 4, Sec. 5). A new derivation of the standard theorem is given (Sec. 6).

II. THEOREM

Consider an infinite homogeneous system in thermal equilibrium, specified by temperature T and chemical potential μ. The two-particle correlation function is defined by

$$G(\mathbf{x}-\mathbf{y})=\langle\rho(\mathbf{x})\rho(\mathbf{y})\rangle-\langle\rho(\mathbf{x})\rangle\langle\rho(\mathbf{y})\rangle, \qquad (1)$$

where $\rho(\mathbf{x})$ is density at position $\mathbf{x}$, and brackets $\langle\ \rangle$ denote thermal averaging. The standard result[2] is that

$$\int d\mathbf{x}\, G(\mathbf{x})=kT\frac{\partial\rho}{\partial\mu}, \qquad (2)$$

where ρ is mean density. An equivalent statement is that in a large subvolume Ω' the fluctuation of particle number

$$N'=\int_{\Omega'} d\mathbf{x}\,\rho(\mathbf{x})$$

is given by

$$\langle N'^2\rangle-\langle N'\rangle^2=\Omega' kT(\partial\rho/\partial\mu), \qquad (3)$$

or with a different form of the right-hand side

$$\langle N'^2\rangle-\langle N'\rangle^2=\Omega'\rho kT(\partial\rho/\partial p), \qquad (4)$$

where p is pressure.

The usual argument is that for large enough Ω' one can ignore interaction across the dividing surface with the remainder of the system. The latter is treated merely as a reservoir of particles. The subsystem in Ω' is then represented, to some unspecified degree of accuracy, by a grand canonical ensemble. Equation (3) is readily derived, and (2) follows from it.

The theorem has been stated for an infinite system. In formal discussion one considers first a large but finite system, of volume Ω. We then use the conventional periodic boundary conditions, so that the quantity on the right-hand side of (1) remains a function only of $(\mathbf{x}-\mathbf{y})$. It is essential that the limit $\Omega\to\infty$ is taken *before* the integration in (2) is performed. It is easily seen that the quantity

$$\lim_{\Omega\to\infty}\int_\Omega d\mathbf{x}\, G(\mathbf{x})$$

is ensemble dependent. In fact, it is proportional to the

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[1] G. Lüders (unpublished).
[2] See for example L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press, London, 1958), p. 365.

mean square fluctuation in total particle number, zero for the canonical ensemble but not for the grand canonical. This ensemble dependence arises through terms in G of the type (constant/Ω), which do not, however, contribute to

$$\int d\mathbf{x} \lim_{\Omega\to\infty} G(\mathbf{x}). \tag{5}$$

It is a consequence of the theorem under discussion that the order of integration and limit can be interchanged for the grand canonical ensemble, but this is not trivial.

Because the exponential factor improves the convergence of the integral, it is plausible that with $\mathbf{k}\neq 0$

$$\int d\mathbf{x}\, e^{-i\mathbf{k}\cdot\mathbf{x}} \lim_{\Omega\to\infty} G(\mathbf{x}) = \lim_{\Omega\to\infty} \int_\Omega e^{-i\mathbf{k}\cdot\mathbf{x}} G(\mathbf{x}).$$

Denoting by $\tilde{G}(\mathbf{k})$ the Fourier transform of G, the theorem can then be restated as

$$\lim_{\mathbf{k}\to 0}\{\lim_{\Omega\to\infty} \tilde{G}(\mathbf{k})\} = kT\frac{\partial\rho}{\partial\mu}. \tag{6}$$

In this form it has considerable importance in scattering problems.[3]

III. BCS MODEL[4]

We are not concerned here with complications arising from Coulomb forces, electron lattice interaction, etc. We consider only the simple model in which spin-$\frac{1}{2}$ particles interact through local potentials of finite range. In second quantization, the density is given in terms of field operations by

$$\rho(\mathbf{x}) = \psi_1^\dagger(\mathbf{x})\psi_1(\mathbf{x}) + \psi_2^\dagger(\mathbf{x})\psi_2(\mathbf{x}),$$

where suffices indicate spin states. With periodic boundary conditions on the total volume Ω, we have

$$\psi_1 = \textstyle\sum_k a(k)\Omega^{-1/2} e^{i\mathbf{k}\cdot\mathbf{x}},$$
$$\psi_2 = \textstyle\sum_k b(k)\Omega^{-1/2} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

The particle absorption operators a and b are in turn related to Valatin-Boguliubov[5] quasi-particle operators α and β by

$$a(\mathbf{k}) = u(\mathbf{k})\alpha(\mathbf{k}) + v(\mathbf{k})\beta^\dagger(-\mathbf{k}),$$
$$b(\mathbf{k}) = u(\mathbf{k})\beta(\mathbf{k}) - v(\mathbf{k})\alpha^\dagger(-\mathbf{k}),$$

where

$$|u(\mathbf{k})|^2 + |v(\mathbf{k})|^2 = 1.$$

About the function u and v we need only note here that v approaches zero above the Fermi surface, and unity below it; only in the transition region are *both* u and v substantially different from zero. The ground state is taken to be the vacuum of quasi-particles. For finite temperatures a noninteracting Fermi gas of quasi-particles is supposed. In either case expectation values of products of field operators can be evaluated by Wick's pairing theorem or its generalization.[6]

Then

$$\begin{aligned} G(\mathbf{x}-\mathbf{y}) &= \sum_{ij}\{\langle\psi_i^\dagger(\mathbf{x})\psi_i(\mathbf{x})\psi_j^\dagger(\mathbf{y})\psi_j(\mathbf{y})\rangle \\ &\qquad - \langle\psi_i^\dagger(\mathbf{x})\psi_i(\mathbf{x})\rangle\langle\psi_j^\dagger(\mathbf{y})\psi_j(\mathbf{y})\rangle\} \\ &= \sum_{ij}\{\langle\psi_i^\dagger(\mathbf{x})\psi_j(\mathbf{y})\rangle\langle\psi_i(\mathbf{x})\psi_j^\dagger(\mathbf{y})\rangle \\ &\qquad - \langle\psi_i^\dagger(\mathbf{x})\psi_j^\dagger(\mathbf{y})\rangle\langle\psi_i(\mathbf{x})\psi_j(\mathbf{y})\rangle\} \\ &= \sum_{ij}\{\langle\psi_i^\dagger(\mathbf{x})\psi_j(\mathbf{y})\rangle\delta_{ij}\delta(\mathbf{x}-\mathbf{y}) \\ &\quad - \langle\psi_i^\dagger(\mathbf{x})\psi_j(\mathbf{y})\rangle\langle\psi_j^\dagger(\mathbf{y})\psi_i(\mathbf{x})\rangle \\ &\qquad + \langle\psi_i^\dagger(\mathbf{x})\psi_j^\dagger(\mathbf{y})\rangle\langle\psi_j(\mathbf{y})\psi_i(\mathbf{x})\rangle\}, \end{aligned}$$

where first the pairing theorem and then the anticommutation rules have been used. Finally[5]

$$G(\mathbf{x}-\mathbf{y}) = \rho\delta(\mathbf{x}-\mathbf{y}) - 2|h(\mathbf{x}-\mathbf{y})|^2 + 2|\chi(\mathbf{x}-\mathbf{y})|^2, \tag{7}$$

where

$$h(\mathbf{x}-\mathbf{y}) = \langle\psi_1^\dagger(\mathbf{y})\psi_1(\mathbf{x})\rangle = \langle\psi_2^\dagger(\mathbf{y})\psi_2(\mathbf{x})\rangle,$$
$$\chi(\mathbf{x}-\mathbf{y}) = \langle\psi_1(\mathbf{y})\psi_2(\mathbf{x})\rangle = -\langle\psi_2(\mathbf{y})\psi_1(\mathbf{x})\rangle.$$

The Fourier transforms of h and χ can be expressed in terms of u and v:

$$\tilde{h}(\mathbf{k}) = \langle a^\dagger(\mathbf{k})a(\mathbf{k})\rangle = |v|^2\{1-f\} + |u|^2 f,$$
$$\tilde{\chi}(\mathbf{k}) = \langle a(\mathbf{k})b(-\mathbf{k})\rangle = u(\mathbf{k})v(\mathbf{k})\{2f-1\},$$

where $f(\mathbf{k})$ is the occupation probability for quasi-particle states of momentum $\mathbf{k}$. Note that $0\leqslant f\leqslant 1$ and $f\to 0$ as $T\to 0$. The Fourier transform of G can then be computed, and one obtains[7]

$$\lim_{k\to 0}\{\lim_{\Omega\to\infty}\tilde{G}(\mathbf{k})\} = 2\int\frac{d\mathbf{k}}{(2\pi)^3} \{f[1-f] + |uv|^2(1+[1-2f]^2)\}. \tag{8}$$

For an ideal gas $uv=0$. It is then readily verified that (8) has the value (6). But when pairing is introduced the theorem is no longer satisfied. To see this it is

[3] See for example the reviews of J. de Boer, *Reports on Progress in Physics* (The Physical Society, London, 1948–49), Vol. 12 and L. Van Hove and K. W. McVoy Nucl. Phys. **33**, 468 (1962).

[4] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).

[5] J. G. Valatin, Nuovo Cimento **7**, 843 (1958). N. N. Boguliubov *ibid.* **7**, 794 (1958).

[6] C. Bloch and C. DeDominicis, Nucl. Phys. **7**, 459 (1958). M. Gaudin, *ibid.* **15**, 89 (1960).

[7] It happens that the order of limits does not matter here. It *would* matter, in the ground-state ($T=0$) problem, for example, if the BCS wave function were replaced by its projection on to a definite particle number. However, with the limits taken in the proper order the result should not depend on such a change. This is confirmed in the particular case of the so-called strong coupling model, by work of Mittelstaedt (to be published).

sufficient to note that (8) does not behave as required as $T\rightarrow 0$. The pairing does not much change ρ as a function of μ, and in particular $(kT\partial\rho/\partial\mu)$ still goes to zero at zero temperature. But the $|uv|^2$ term of (8) remains finite when $T\rightarrow 0$, i.e., when $f\rightarrow 0$.

IV. REDUCED HAMILTONIAN

We have to revise either the statistical mechanical argument or the BCS model. It is first necessary to dispose of the fact that, according to Bogoliubov[8] the BCS method gives essentially the *exact* solution for the so called "reduced Hamiltonian." The "reduction" process is the following. Start with, say, an ordinary local interaction energy

$$\frac{1}{2}\int d\mathbf{x}dy\, \psi_1^\dagger(\mathbf{x})\psi_2^\dagger(\mathbf{y})v(\mathbf{x}-\mathbf{y})\psi_2(\mathbf{y})\psi_1(\mathbf{x}).$$

In the Fourier decomposition

$$\psi_1^\dagger(\mathbf{x})\psi_2^\dagger(\mathbf{y})=\sum_{k_1k_2} a^\dagger(\mathbf{k}_1)b^\dagger(\mathbf{k}_2)e^{-i(\mathbf{k}_1\cdot\mathbf{x}+\mathbf{k}_2\cdot\mathbf{y})}$$

drop all terms with nonzero total momentum $(\mathbf{k}_1+\mathbf{k}_2)$. In configuration space the resulting reduced interaction is

$$\frac{1}{2\Omega}\int d\mathbf{x}dydz\, \psi_1^\dagger(\mathbf{x}+\mathbf{z})\psi_2^\dagger(\mathbf{x}+\mathbf{z})v(\mathbf{x}-\mathbf{y})\psi_2(\mathbf{y})\psi_1(\mathbf{x}). \quad (9)$$

Now this interaction picks up a pair of particles in one region of space and takes it equally to any other region without respect for distance. So one certainly cannot regard any subvolume of the system as even approximately isolated. The failure of the standard theorem with the reduced Hamiltonian is therefore no mystery. Moreover, because of the unphysical nature of the reduced Hamiltonian, any result which depended essentially on its existence and solubility should not be trusted.

V. MEISSNER EFFECT, AND THE ZERO-TEMPERATURE CASE

The nonlocality of the reduced Hamiltonian was the cause of an earlier difficulty in the BCS theory, the lack of gauge invariance for the Meissner effect.[9] It turns out that this is closely connected with the present problem. We consider here only the case of zero temperature, i.e., the ground state.

In the presence of a vector potential A the current has the form

$$\mathbf{J}(\mathbf{x})=\mathbf{j}(\mathbf{x})-\rho(\mathbf{x})\mathbf{A}(\mathbf{x}),$$

where $\mathbf{j}$ does not depend explicitly on $\mathbf{A}$. To first order in $\mathbf{A}$ the expectation value of the current is

$$\langle\mathbf{J}(\mathbf{x})\rangle=-\rho\mathbf{A}(\mathbf{x})-\sum_{n\neq0}(E_0-E_n)^{-1}\times\left\{\langle0|\mathbf{j}(\mathbf{x})|n\rangle\left\langle n\left|\int dy\,\mathbf{j}(\mathbf{y})\mathbf{A}(\mathbf{y})\right|0\right\rangle+\left\langle0\left|\int dy\,\mathbf{j}(\mathbf{y})\mathbf{A}(\mathbf{y})\right|n\right\rangle\langle n|\mathbf{j}(\mathbf{x})|0\rangle\right\}. \quad (10)$$

If for the ground state $|0\rangle$ and excited states $|n\rangle$ the states of BCS theory are used, the last two terms are found to be negligible for slowly varying fields $\mathbf{A}$. One finds then simply

$$\langle\mathbf{J}(\mathbf{x})\rangle=-\rho\mathbf{A}(\mathbf{x}). \quad (11)$$

Unfortunately, the divergence of this current is not in general zero, and the current does not vanish for a fictitious potential $\mathbf{A}=\nabla\chi$. These defects can be blamed on the nonlocality of the reduced Hamiltonian, because of which the current is not conserved. They are removed for example in the more elaborate "generalized random phase approximation"[8] (GRPA) and equivalent approaches.[10] In these the whole of the Hamiltonian is allowed for, in an approximate but gauge-invariant way. The result (11) is then replaced by, in momentum space and for small $\mathbf{k}$ (London limit),

$$\langle\tilde{J}_\mu(\mathbf{k})\rangle=\rho\left(\delta_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right)\tilde{A}_\nu(\mathbf{k}).$$

Here the offending longitudinal part has been removed.

In the GRPA the Meissner effect is gauge invariant and matrix elements of the current obey the continuity equation. An important sum rule[9] (which follows in general from commutation rules and continuity) is then respected. This proves very relevant to our problem. Note, however, that we are not committed in the following reasoning to the details of that particular approximation, the GRPA, which has difficulties of its own.[11]

Consider the current induced by a fictitious static field $\mathbf{A}=\nabla\chi$. Its divergence should certainly vanish. If we substitute $\mathbf{A}=\nabla\chi$ in (10), partially integrate with respect to y, and use the continuity equation

$$\begin{aligned}\langle n|\nabla\cdot\mathbf{j}|m\rangle&=-\langle n|\dot{\rho}|m\rangle\\&=-i(E_n-E_m)\langle n|\rho|m\rangle,\end{aligned} \quad (13)$$

we obtain the identity

$$0=\rho\nabla^2\chi(x)-\sum(E_0-E_n)\times\left\{\langle0|\rho(\mathbf{x})|n\rangle\left\langle n\left|\int dy\rho(\mathbf{y})\chi(\mathbf{y})\right|0\right\rangle+\left\langle0\left|\int dy\rho(\mathbf{y})\chi(\mathbf{y})\right|n\right\rangle\langle n|\rho(\mathbf{x})|0\rangle\right\}.$$

[8] N. N. Bogoliubov, Suppl. Physica **26**, 1 (1960).

[9] P. W. Anderson, Phys. Rev. **110**, 827 (1958); **112**, 1900 (1958); G. Rickayzen, *ibid.* **115**, 795 (1959).

[10] V. Ambegaoker and L. Kadanoff, in *Notes on the Many-Body Problem*, edited by C. Fronsdal (W. A. Benjamin, Inc., New York, 1962), pp. 66–84. Other literature can be traced from references in this paper.

[11] The quantity $G(\mathbf{x}-\mathbf{y})-\rho\delta(\mathbf{x}-\mathbf{y})+\rho^2$, which should equal $\langle\psi^\dagger(\mathbf{x})\psi^\dagger(\mathbf{y})\psi(\mathbf{y})\psi(\mathbf{x})\rangle$, does not in the GRPA come out positive for small $|\mathbf{x}-\mathbf{y}|$.

From this follows the sum rule

$$\sum_{n\neq 0}(E_0-E_n)\{\langle 0|\rho(\mathbf{x})|n\rangle\langle n|\rho(\mathbf{y})|0\rangle + \langle 0|\rho(\mathbf{y})|n\rangle\langle n|\rho(\mathbf{x})|0\rangle\}=\rho\nabla^2\delta(\mathbf{x}-\mathbf{y}). \quad (14)$$

A weight function σ is defined by

$$\sigma(E,\mathbf{k})=\sum_{n\neq 0}\langle 0|\rho(0)|n\rangle\langle n|\rho(0)|0\rangle \times\delta(E_n-E_0-E)\delta(\mathbf{P}_n-\mathbf{k})(2\pi)^3, \quad (15)$$

where $\mathbf{P}_n$ is momentum of state n. Noting that σ will not depend on the sign of $\mathbf{k}$, the sum rule (14) takes the form

$$\int_0^\infty dE\, E\sigma(E,\mathbf{k})=\tfrac{1}{2}k^2\rho. \quad (16)$$

The result (16) is relevant for the correlation function because

$$\int_0^\infty dE\,\sigma(E,\mathbf{k})=\tilde{G}(\mathbf{k}). \quad (97)$$

If for momentum $\mathbf{k}$ the spectrum of excitations is bounded below by some $E(\mathbf{k})$, from (16) and (17)

$$0\leqslant\tilde{G}(\mathbf{k})\leqslant k^2\rho/2E(\mathbf{k}).$$

Therefore a sufficient condition for

$$\tilde{G}(\mathbf{k})\to 0 \quad\text{as}\quad \mathbf{k}\to 0 \quad (18)$$

is that

$$k^2/E(\mathbf{k})\to 0 \quad\text{as}\quad \mathbf{k}\to 0. \quad (19)$$

In the GRPA the lowest excitations for small k are the sound waves, for which $E(\mathbf{k})\propto|\mathbf{k}|$. Thus condition (19) is met and the result (18) expected from (6) is obtained.

Actually it is somewhat careless to expect (18) at zero temperature on the basis only of (6) at $T\neq 0$ and the assumption that $\partial\rho/\partial\mu$ remains finite at $T=0$. We have now taken the limit $T\to 0$ before the limit $\mathbf{k}\to 0$, and the derivations in either Sec. 2 or Sec. 6 do not directly apply. It is easy to think of functions for which the order of limits would be important. Thus, even when (6) is accepted, some special discussion of the case $T=0$ is necessary, such as that given here.

VI. NONZERO TEMPERATURE

Finally we demonstrate (6) from the continuity equation, and some plausible assumptions, in the case of finite temperature. Here the *total* system is represented by a grand canonical ensemble, with density operator

$$M=\exp[-\beta(H-\mu N)]/\mathrm{Tr}\exp[-\beta(H-\mu N)],$$

where $\beta=1/kT$. It is convenient to consider small *local* variations of μ:

$$M\to\exp-\beta\Big[H-\mu N-\int d\mathbf{y}\,\delta\mu(\mathbf{y})\rho(\mathbf{y})\Big]\Big/\mathrm{Tr}\exp-\beta\Big(H-\mu N-\int d\mathbf{y}\,\delta\mu\rho\Big).$$

This could be effected physically by the application of an external field. It is plausible that for the density at any point a variation $\delta\mu$ which is uniform over a sufficiently large neighborhood is equivalent to a variation of μ for the complete system:

$$\frac{\partial\rho}{\partial\mu}=\int d\mathbf{y}\frac{\delta\langle\rho(\mathbf{x})\rangle}{\delta\mu(\mathbf{y})}=\int d\mathbf{x}\frac{\delta\langle\rho(\mathbf{x})\rangle}{\delta\mu(\mathbf{y})}. \quad (20)$$

This assumption, that the equation of state in a sufficiently large region is independent of conditions outside that region, is already implicit in the argument of Sec. 2. No additional assumption about number fluctuations is made here.

Consider the identity

$$0=\mathrm{Tr}\Big\{[\rho(\mathbf{x})-\langle\rho(\mathbf{x})\rangle]\times\exp-\beta\Big[H-\int d\mathbf{y}\,\mu(\mathbf{y})\rho(\mathbf{y})\Big]\Big\}. \quad (21)$$

Functional differentiation with respect to $\mu(y)$ yields

$$kT[\delta\langle\rho(\mathbf{x})\rangle/\delta\mu(\mathbf{y})]=\langle\rho(\mathbf{x})\rho'(\mathbf{y})\rangle-\langle\rho(\mathbf{x})\rangle\langle\rho(\mathbf{y})\rangle. \quad (22)$$

In classical statistics, where there are no commutation difficulties, $\rho'(\mathbf{x})$ would simply equal $\rho(\mathbf{x})$. Combining (1), (20), and (22), the theorem would be proved. There is no special appeal here to continuity, but continuity is anyway inherent in classical Hamiltonian particle mechanics. In quantum mechanics, because $\rho(\mathbf{x})$ and H do not commute,

$$\rho'(\mathbf{y})=\frac{1}{\beta}\int_0^\beta d\xi\, e^{-\xi H}\rho(\mathbf{y})e^{\xi H}. \quad (23)$$

With the obvious definition of G', one has

$$kT\frac{\partial\rho}{\partial\mu}=\int d\mathbf{x}\,G'(\mathbf{x}).$$

This is not yet the result required.

Let us express the quantities $\langle\rho(\mathbf{x})\rho'(\mathbf{y})\rangle$ and $\langle\rho(\mathbf{x})\rho(\mathbf{y})\rangle$ in terms of the matrix elements of ρ between the eigenstates $|n\rangle$ of energy and momentum:

$$\mathbf{P}|n\rangle=\mathbf{P}_n|n\rangle,\quad (H-\mu N)|n\rangle=E_n|n\rangle.$$

With

$$\sigma(E,\mathbf{k})=\sum_{n\neq m}e^{-\beta E_n}\langle n|\rho(0)|m\rangle\langle m|\rho(0)|n\rangle \times\delta(E_m-E_n-E)\delta(\mathbf{P}_m-\mathbf{P}_n-\mathbf{k})(2\pi)^3, \quad (25)$$

one finds

$$\tilde{G}'(\mathbf{k})-\tilde{G}(\mathbf{k})=\int dE\,\sigma(E,\mathbf{k})\Big(\frac{1-e^{-\beta E}}{\beta E}-1\Big). \quad (26)$$

At high temperature (the classical case) $\beta\to 0$ and G'

becomes equal to G. We have to show that this is true in general when $\mathbf{k}\rightarrow 0$. That is that for decreasing $\mathbf{k}$ the weight function σ is restricted to energies decreasing to zero.

Consider now the current correlation

$$D_{\mu\nu}(\mathbf{x}-\mathbf{y})=\langle j_\mu(\mathbf{x})j_\nu(\mathbf{y})\rangle. \tag{27}$$

From the continuity equation we have

$$\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial y_\nu}D_{\mu\nu}(\mathbf{x}-\mathbf{y})=\left\langle\frac{\partial\rho(\mathbf{x})}{\partial t}\frac{\partial\rho(\mathbf{y})}{\partial t}\right\rangle,$$

where

$$k_\mu k_\nu\tilde{D}_{\mu\nu}(\mathbf{k})=-\int dE\,E^2\sigma(E,\mathbf{k}). \tag{28}$$

Assuming that $D_{\mu\nu}$ has finite range, so that $\tilde{D}_{\mu\nu}$ is finite at $\mathbf{k}=0$, we have

$$\lim_{\mathbf{k}\to 0}\int dE\,E^2\sigma(E,\mathbf{k})=0,$$

i.e., for small $\mathbf{k}$ the spectral function is concentrated at small E. Assuming that $G(\mathbf{x})$ also has finite range, we obtain

$$\lim_{\mathbf{k}\to 0}\int dE\,\sigma(E,\mathbf{k})=\int G(\mathbf{x})d\mathbf{x}=\text{finite constant}.$$

From (29) and (30), and because σ is positive,

$$\lim_{\mathbf{k}\to 0}\int dE\,\sigma(E,\mathbf{k})\left(\frac{1-e^{-\beta E}}{\beta E}-1\right)=0,$$

or

$$\lim_{\mathbf{k}\to 0}[\tilde{G}'(\mathbf{k})-G(\mathbf{k})]=0.$$

This is the required result.

VII. CONCLUSION

It has been seen that the failure of the BCS theory to give the expected result is related to other defects of that theory, notably the lack of current continuity. Moreover a derivation of the standard theorem has been outlined which seems to rest on weaker and more explicit assumptions than the usual version. Even those who cannot doubt the usual reasoning may attach some value to the new account, because it shows that in any model not satisfying the theorem one or more of several other expected properties cannot be realized. Neither the new nor the old accounts apply directly at zero temperature; the behavior there has been related separately to the excitation spectrum of the ground state.

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